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Uniformization of Simply-Connected Ramified Coverings of the Sphere by Rational Functions

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Abstract—We deduce a system of ODEs describing behavior of critical points and poles of a smooth one-parametric family of rational functions uniformizing a given family of ramified coverings of the Riemann sphere.

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1. INTRODUCTION

Consider a simply-connected compact Riemann surface S which is a ramified covering of the Riemann sphere $\overline{\mathbb{C}}$. It is well known that there exists a rational function R uniformizing the surface. An important problem is to develop approximate methods to find the function. Here we suggest a method based on investigation of dependence of its critical points and poles on the critical values.

Let S have p sheets and $N+1$ points of multiplicities $n_1, n_2, \dots, n_{N+1}$ lie over the infinity. Let we have M branch points of multiplicities $m_1, m_2, \dots, m_M$ over the finite plane; we denote by $A_1, A_2, \dots, A_M$ their projections on the plane.

We note that, as a rule, by given A_1, $A_2, \dots, A_M$ and multiplicities m_1, $m_2, \dots, m_M$, n_1, $n_2, \dots, n_{N+1}$ the surface S is defined not by a unique way. Hurwitz was the first who payed attention to this fact. He posed the problem to determine the number of nonequivalent coverings with a given branch type and suggested some ways to investigate the problem [1, 2]. His investigations were developed by many well-known mathematician, in particular, by H. Weyl [3]. Among the works devoted to the study of the problem we should note the papers by Mednykh [4–6] who obtained effective formulas for the number of nonequivalent coverings with the same branch data. Some results on solving the problem can be found in [7] where some additional bibliography is given. There are also some geometric approaches based on the theory of vector bundles (see, e.g., the survey [8], and the monograph [9]). In spite of the non-uniqueness, mentioned above, for fixed branch multiplicities $m_1, m_2, \dots, m_M$ and $n_1, \dots, n_{N+1}$ we can consider the numbers $A_1, A_2, \dots, A_M$ as local coordinates in the corresponding space of branched coverings.

The essence of our approximate method to determine a rational functions R, uniformizing a given branched covering S of the Riemann sphere, is in the following. Consider the set $\mathfrak{S}$ of all compact Riemann surfaces of genus 0 over the sphere with branch type equivalent to the branch type of S. (This means that the numbers m_k, $1 \le k \le M$ and n_j, $1 \le j \le N+1$, characterizing multiplicities over the finite points and the infinity, for S and for arbitrary element $S' \in \mathfrak{S}$ coincide.) Denote by $\pi : \mathfrak{S} \to \mathbb{C}^M$ the map relating to a surface $S' \in \mathfrak{S}$ the projections A'_j of its branch points lying in the finite part of $\overline{\mathbb{C}}$.

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